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On a divisor problem related to the Epstein zeta-function, II [☆]

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ABSTRACT

Recently by using the theory of modular forms and the Riemann zeta-function, Lü improved the estimates for the error term in a divisor problem related to the Epstein zeta-function established by Sankaranarayanan. In this short note, we are able to further sharpen some results of Sankaranarayanan and of Lü, and to establish corresponding Ω-estimates.

1. Introduction

For a positive definite quadratic form $Q(\mathbf{y}) = Q(y_1, \dots, y_\ell)$ in $\ell \geqslant 2$ variables with integral coefficients, we can write it in Siegel's notation as

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$$Q(\mathbf{y}) = \frac{1}{2}\mathbf{A}[\mathbf{y}] = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y} = \sum_{i<j} a_{ij} y_i y_j + \frac{1}{2}\sum_i a_{ii} y_i^2,$$

where $\mathbf{y}^{\mathrm{t}}$ is the transpose of $\mathbf{y}$, and the matrix $\mathbf{A} = (a_{ij})$ has integral entries which are even on the diagonal, i.e., $a_{ii} \equiv 0 \pmod 2$ for $0 \leqslant i \leqslant \ell$. Then the corresponding Epstein zeta-function is initially defined by the Dirichlet series

$$Z_Q(s) := \sum_{\substack{y_1 \in \mathbb{Z} \\ (y_1,\dots,y_\ell) \neq (0,\dots,0)}} \cdots \sum_{y_\ell \in \mathbb{Z}} Q(y_1, \dots, y_\ell)^{-s} \tag{1.1}$$

for $\Re e\, s > \ell/2$. We can also rewrite it, in the same region, as

$$Z_Q(s) = \sum_{n \geqslant 1} a_n n^{-s},$$

where a_n is the number of the solutions of the equation $Q(\mathbf{y}) = n$ with $\mathbf{y} \in \mathbb{Z}^\ell$. It is known that $Z_Q(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ with only a simple pole at $s = \ell/2$, and satisfies the functional equation of Riemann type

$$\big(d^{1/\ell}/2\pi\big)^s \Gamma(s) Z_Q(s) = \big(d^{1-1/\ell}/2\pi\big)^{\ell/2-s} \Gamma(\ell/2 - s) Z_{\overline{Q}}(\ell/2 - s) \quad (s \in \mathbb{C}),$$

where d is the discriminant of Q and $\overline{Q}(\mathbf{y}) := \frac{1}{2}\mathbf{y}^{\mathrm{t}}(d\mathbf{A}^{-1})\mathbf{y}$ (cf. [9]).

If we write for any integer $k \geqslant 1$,

$$Z_Q(s)^k = \sum_{n \geqslant 1} a_k(n) n^{-s},$$

then

$$a_k(n) = \sum_{n_1 \cdots n_k = n} a_{n_1} \cdots a_{n_k}.$$

In particular $a_1(n) = a_n$. It seems interesting to study the asymptotic behavior of the sum $\sum_{n \leqslant x} a_k(n)$. It is easy to show that its main term is

$$\operatorname*{Res}_{s=\ell/2}\big(Z_Q(s)^k x^s s^{-1}\big) = x^{\ell/2} P_k(\log x),$$

where $P_k(t)$ is a polynomial in t of degree $k-1$. Then the real hard work is to study the error term

$$\Delta_k^*(Q, x) := \sum_{n \leqslant x} a_k(n) - x^{\ell/2} P_k(\log x). \tag{1.2}$$

In 1912, Landau [7] proved that for $\ell = 2$, $\Delta_1^*(Q, x) \ll x^{1/3+\varepsilon}$, where and throughout this paper ε denotes an arbitrarily small positive constant. Landau's method can also be applied to treat the general case. In fact his method implies that for $k \geqslant 1$ and $\ell \geqslant 2$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2 - \ell/(k\ell+1)+\varepsilon}.$$

Later Chandrasekharan and Narasimhan [1] were able to delete the ε in the exponent of x. In [9], Sankaranarayanan improved these classical results by showing that for $k \geqslant 2$ and $\ell \geqslant 3$,

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1/k+\varepsilon}. \tag{1.3}$$

Recently inspired by Iwaniec's book [5], Lü [8] was able to improve (1.3) for the quadratic forms of level one (see [5, Chapter 11]). These quadratic forms are defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{A}[\mathbf{y}]$ with $\mathrm{diag}(\mathbf{A}) = \mathrm{diag}(\mathbf{A}^{-1}) \equiv 0 \pmod 2$, where $\mathrm{diag}(\mathbf{A})$ denotes the set of entries on the diagonal of the matrix $\mathbf{A}$. Moreover we have that $\det(\mathbf{A}) = 1$, $\mathbf{A}$ is equivalent to $\mathbf{A}^{-1}$, and the number of variables satisfies $\ell \equiv 0 \pmod 8$. Denote by $\mathcal{Q}_\ell$ the set of quadratic forms of level one with ℓ variables. For $Q \in \mathcal{Q}_\ell$, we have (see [5, (11.32)] or [8, Lemma 2.1])

$$a_n = A_\ell \sigma_{\ell/2-1}(n) + a_f(n, Q) \quad (n \geqslant 1),$$

where

$$A_\ell := \frac{(2\pi)^{\ell/2}}{\zeta(\ell/2)\Gamma(\ell/2)}, \qquad \sigma_k(n) = \sum_{d|n} d^k,$$

$\zeta(s)$ is the Riemann zeta-function, $\Gamma(s)$ is the Gamma function and $a_f(n, Q)$ is the nth Fourier coefficient of a cusp form $f(z, Q)$ of weight $\ell/2$ with respect to the full modular group $\mathrm{SL}(2, \mathbb{Z})$. Thus

$$Z_Q(s) = A_\ell \zeta(s - \ell/2 + 1)\zeta(s) + L(s, f) \quad (\Re e\, s > \ell/2), \tag{1.4}$$

where $L(s, f)$ is the Hecke L-function associated with $f(z, Q)$. According to the well known Deligne's work [2], we have

$$\left|a_f(n, Q)\right| \leqslant n^{(\ell/2-1)/2}\tau(n), \tag{1.5}$$

where $\tau(n)$ is the divisor function. With the help of these properties, Lü proved, by complex integration method, a better estimate than Sankaranarayanan's (1.3) for all $k \geqslant 3$ and $8 \mid \ell$. For $r \geqslant 0$, the r-dimensional divisor function $\tau_r(n)$ is defined by

$$\zeta(s)^r = \sum_{n \geqslant 1} \tau_r(n) n^{-s} \quad (\Re e\, s > 1).$$

The r-dimensional divisor problem concerns the estimate of the error term

$$\Delta_r(x) := \sum_{n \leqslant x} \tau_r(n) - \operatorname*{Res}_{s=1}\bigl(\zeta(s)^r x^s s^{-1}\bigr) = \sum_{n \leqslant x} \tau_r(n) - x G_r(\log x), \tag{1.6}$$

where $G_r(t)$ is a polynomial of degree $r-1$ if $r \geqslant 1$ and $G_0(t) \equiv 0$. It is known that

$$\Delta_r(x) \ll x^{\theta_r+\varepsilon} \quad (x \geqslant 2) \tag{1.7}$$

where

$$\theta_0 = 0, \qquad \theta_1 = 0, \qquad \theta_2 = 131/416, \qquad \theta_3 = 43/96 \tag{1.8}$$

and

$$\theta_r = \begin{cases} (3r-4)/(4r), & \text{if } 4 \leqslant r \leqslant 8, \\ 35/54, & \text{if } r = 9, \\ 41/60, & \text{if } r = 10, \\ 7/10, & \text{if } r = 11, \\ (r-2)/(r+2), & \text{if } 12 \leqslant r \leqslant 25, \\ (r-1)/(r+4), & \text{if } 26 \leqslant r \leqslant 50, \\ (31r-98)/(32r), & \text{if } 51 \leqslant r \leqslant 57, \\ (7r-34)/(7r), & \text{if } r \geqslant 58. \end{cases} \tag{1.9}$$

(The case of $r = 0, 1$ is trivial. See [3] for $r = 2$, [6] for $r = 3$ and [4, Theorem 13.2] for $r \geqslant 4$.) Lü's result (see [8, Theorem 1.2]) can be stated as follows

$$\Delta_k^*(Q, x) \ll \begin{cases} x^{\ell/2-1/2+\varepsilon}, & \text{if } k = 3, \\ x^{\ell/2-1+\theta_k+\varepsilon}, & \text{if } k \geqslant 4. \end{cases} \tag{1.10}$$

In this short note, we can further improve Sankaranarayanan's (1.3) with $k = 2$ and Lü's (1.10) with $k = 3$.

Theorem 1. *Let $k \geqslant 2$ and $8 \mid \ell$. Then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$, we have*

$$\Delta_k^*(Q, x) \ll x^{\ell/2-1+\theta_k+\varepsilon},$$

where θ_k is the exponent in (1.7).

For comparison, we note that

$$\ell/2 - 1 + \theta_k = \begin{cases} \ell/2 - 1/2 - 0.185\ldots & \text{if } k = 2, \\ \ell/2 - 1/2 - 0.052\ldots & \text{if } k = 3, \end{cases}$$

which are better than (1.3) with $k = 2$ and (1.10), respectively.

For $k = 2$ or 3, we also can establish Ω-type result.

Theorem 2. *Let $2 \leqslant k \leqslant 8$ and $8 \mid \ell$. If there is a positive constant δ such that*

$$\theta_r \leqslant (k-1)/(2k) - \delta \quad (0 \leqslant r \leqslant k-1), \tag{1.11}$$

then for any quadratic form $Q(\mathbf{y}) \in \mathcal{Q}_\ell$ and $\varepsilon > 0$, we have

$$\Delta_k^*(Q, x) = \Omega\left(x^{\ell/2-1+(k-1)/(2k)} (\log x)^{(k-1)/(2k)} (\log_2 x)^{\beta_k} (\log_3 x)^{-\gamma_k-\varepsilon}\right) \tag{1.12}$$

where $\beta_k := (k^{(2k)/(k+1)} - 1)(k+1)/(2k)$ and $\gamma_k := (3k-1)/(4k)$.

In particular (1.12) *holds unconditionally for $k = 2$ or 3.*

Our method is different from [8]. First we shall establish relations between $\Delta_k(x)$ and $\Delta_k^*(Q, x)$ and then deduce Theorems 1 and 2 from known O-type and Ω-type estimates for $\Delta_k(x)$.

2. Preliminary lemmas

This section is devoted to establish three preliminary lemmas, which will be needed in the proof of Theorems 1 and 2.

Lemma 2.1. *For any $\varepsilon > 0$, we have*

$$\int_1^x \Delta_r(t)\,\mathrm{d}t \ll_{r,\varepsilon} x^{1+\delta_r+\varepsilon} \quad (x \geqslant 1),$$

where

$$\delta_r := \begin{cases} 1/2 - 1/r, & \text{if } r = 2, 4, 6, 8, \\ 1/2 - 1/(r+1), & \text{if } r = 1, 3, 5, 7. \end{cases} \tag{2.1}$$

Proof. By Perron's formula [11, Theorem II.2.3], we obtain, with $b := 1 + 1/\log x$,

$$\int_0^x \Delta_r(u)\,\mathrm{d}u = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s - \int_0^x u G_r(\log u)\,\mathrm{d}u, \tag{2.2}$$

where $b := 1 + 1/\log x$ and $F_r(s;x) := \zeta(s)^r x^{s+1}/\{s(s+1)\}$.

Let $\max\{1 - 6/r, 0\} < a < 1$. By using the classical estimate

$$\zeta(s) \ll \big(|t| + 2\big)^{\max\{(1-\sigma)/3,0\}} \log\big(|t| + 2\big),$$

we deduce that for all $\varepsilon > 0$ and $T > 0$,

$$\int_{a \leqslant \sigma \leqslant b,\ |\tau| = T} \big|F_r(s;x)\big| |\mathrm{d}s| \ll \big(x^2 T^{-2} + x^{1+a} T^{-2+\max\{(1-a)r/3,0\}}\big)(\log T)^r,$$

and

$$\int_{\sigma = b,\ |t| \geqslant T} \big|F_r(s;x)\big| |\mathrm{d}s| \ll x^2 T^{-1} (\log T)^r.$$

Using the preceding estimates and shifting the line of integration from $\sigma = b$ to $\sigma = a$, the residue theorem implies that

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s &= \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} F_r(s;x)\,\mathrm{d}s + O\left(\frac{x^2}{T}(\log T)^r\right) \\ &= \int_0^x u G_r(u)\,\mathrm{d}u + \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}T}^{a+\mathrm{i}T} F_r(s;x)\,\mathrm{d}s \\ &\quad + O\left(\frac{x^2}{T}(\log T)^r + \frac{x^{1+a}}{T^{2-\max\{(1-a)r/3,0\}}}(\log T)^r\right), \end{aligned}$$

where we have used the relation

$$\operatorname*{Res}_{s=1}\big(F_r(s;x)\big) = \int_0^x uG_r(u)\,\mathrm{d}u.$$

Making $T \to \infty$ and inserting the obtained formula into (2.2), we find that

$$\begin{aligned}\int_1^x \Delta_r(u)\,\mathrm{d}u &= \frac{1}{2\pi \mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} F_r(s;x)\,\mathrm{d}s \\ &\ll x^{1+a}\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^r}{(|t|+1)^2}\,\mathrm{d}t.\end{aligned}$$

When $r = 2, 4, 6, 8$, the last integral is convergent for any $a > 1/2 - 1/r \geqslant \max\{1 - 6/r, 0\}$ (see [4, Lemma 13.1 and Theorem 13.4]). For $r = 2k - 1$ ($1 \leqslant k \leqslant 4$), we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^r}{(|t|+1)^2}\,\mathrm{d}t &\leqslant \left\{\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^{2(k-1)}}{(|t|+1)^2}\,\mathrm{d}t\right\}^{1/2}\left\{\int_{-\infty}^{+\infty}\frac{|\zeta(a+\mathrm{i}t)|^{2k}}{(|t|+1)^2}\,\mathrm{d}t\right\}^{1/2} \\ &< \infty\end{aligned}$$

provided $a > 1/2 - 1/(2k) = 1/2 - 1/(r+1) \geqslant \max\{1 - 6/r, 0\}$. This completes the proof. □

Lemma 2.2. *For $r \geqslant 0$, we have*

$$\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} = x^{\ell/2}G_r^*(\log x) + x^{\ell/2-1}\Delta_r(x) + O\big(x^{\ell/2-1+\delta_r}\big), \tag{2.3}$$

where $G_r^*(t)$ is a polynomial of degree $r-1$ with the convention that $G_0^*(t) \equiv 0$ and the constant $\delta_r \geqslant 0$ is given by (2.1)*. In particular*

$$\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} = x^{\ell/2}G_r^*(\log x) + O\big(x^{\ell/2-1+\theta_r}\big). \tag{2.4}$$

Proof. With the help of (1.6) and Lemma 2.1, a simple partial summation yields

$$\begin{aligned}\sum_{n\leqslant x}\tau_r(n)n^{\ell/2-1} &= \int_1^x t^{\ell/2-1}\big(tG_r(\log t)\big)'\,\mathrm{d}t + \int_{1-}^x t^{\ell/2-1}\,\mathrm{d}\Delta_r(t) \\ &= x^{\ell/2}G_r^*(\log x) + x^{\ell/2-1}\Delta_r(x) + O\big(x^{\ell/2-1+\delta_r}\big).\end{aligned}$$

This completes the proof. □

In order to state our third lemma, it is necessary to introduce some notation.

By (1.4), we can write, for $\Re e\, s > \ell/2$,

$$Z_Q(s)^k = \sum_{0\leqslant r\leqslant k} A_\ell^r C_k^r \zeta(s)^r L(s,f)^{k-r}\zeta(s-\ell/2+1)^r,$$

$$\zeta(s-\ell/2+1)^k = \sum_{0\leqslant r\leqslant k} A_\ell^{-k} C_k^r (-1)^{k-r}\zeta(s)^{-k} L(s,f)^{k-r} Z_Q(s)^r.$$

These imply that

$$a_k(n) = \sum_{0\leqslant r\leqslant k} A_\ell^r C_k^r \sum_{dm=n} b_{k,r}(d)\tau_r(m)m^{\ell/2-1}, \tag{2.5}$$

$$\tau_k(n)n^{\ell/2-1} = \sum_{0\leqslant r\leqslant k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{dm=n} c_{k,r}(d)a_r(m), \tag{2.6}$$

where $b_{k,r}$ and $c_{k,r}$ are defined by the relations

$$\zeta(s)^r L(s,f)^{k-r} = \sum_{n\geqslant 1} b_{k,r}(n)n^{-s}, \qquad \zeta(s)^{-k} L(s,f)^{k-r} = \sum_{n\geqslant 1} c_{k,r}(n)n^{-s},$$

for $\Re e\, s > \ell/2$.

Lemma 2.3. *Let $j\geqslant 0$, $k\geqslant 2$, $0\leqslant r\leqslant k$, $8\mid\ell$ and $\theta > (\ell+2)/4$. Then for any quadratic form $Q(\mathbf{y})\in\mathcal{Q}_\ell$ and $d_{k,r} = b_{k,r}$ or $c_{k,r}$, we have*

$$\sum_{n\leqslant x} \frac{|d_{k,r}(n)|}{n^\theta} \ll_{j,\ell,\theta} 1 \quad (x\geqslant 2), \tag{2.7}$$

$$\sum_{n\leqslant x} \frac{d_{k,r}(n)(\log n)^j}{n^\theta} = C_f(j,k,r,\theta) + O\left(x^{-\theta+(\ell+2)/4+\varepsilon}\right) \quad (x\geqslant 2), \tag{2.8}$$

where $C_f(j,k,r,\theta)$ is a constant.

Proof. By the definition of $b_{k,r}$ and $c_{k,r}$, we have

$$b_{k,r}(n) = \sum_{d_1\cdots d_r m_1\cdots m_{k-r}=n} a_f(Q,m_1)\cdots a_f(Q,m_{k-r}),$$

$$c_{k,r}(n) = \sum_{d_1\cdots d_k m_1\cdots m_{k-r}=n} \mu(d_1)\cdots\mu(d_k)a_f(Q,m_1)\cdots a_f(Q,m_{k-r}).$$

We treat only the case of $b_{k,r}$ and the latter is completely similar. With the help of the Deligne inequality (1.5), we have

$$\begin{aligned}\sum_{n\leqslant x}\left|b_{k,r}(n)\right| &\leqslant \sum_{d\leqslant x}\tau_r(d)\sum_{m\leqslant x/d}\tau_{2(k-r)}(m)m^{(\ell-2)/4}\\ &\leqslant \sum_{d\leqslant x}\tau_r(d)(x/d)^{\ell/4+1/2}(\log x)^{2k-2r-1}\\ &\ll_{j,\ell} x^{(\ell+2)/4}(\log x)^{2k-2r-1} \quad (0\leqslant r\leqslant k).\end{aligned}$$

From this, a simple partial integration allows us to deduce (2.7) and (2.8). $\square$

3. Proof of Theorem 1

By (2.5), (2.4) of Lemma 2.2 and (2.7) of Lemma 2.3, it follows that

$$\sum_{n\leqslant x} a_k(n) = x^{\ell/2} \sum_{0\leqslant r\leqslant k} C_k^r A_\ell^r \sum_{d\leqslant x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*\big(\log(x/d)\big) + O\big(x^{\ell/2-1+\theta_k+\varepsilon}\big).$$

Since $\ell/2 > (\ell+2)/4$, (2.8) of Lemma 2.3 implies that

$$\sum_{0\leqslant r\leqslant k} C_k^r A_\ell^r \sum_{d\leqslant x} \frac{b_{k,r}(d)}{d^{\ell/2}} G_r^*\big(\log(x/d)\big) = P_k(\log x) + O\big(x^{1/2-\ell/4+\varepsilon}\big).$$

Inserting it into the preceding formula, we get the required result. $\square$

4. Proof of Theorem 2

From (2.6) and (1.2), we can deduce that

$$\sum_{n\leqslant x} \tau_k(n) n^{\ell/2-1} = x^{\ell/2} G_k^*(\log x) + O\big(x^{(\ell+2)/4+\varepsilon}\big)$$
$$+ \sum_{0\leqslant r\leqslant k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d) \Delta_r^*(Q, x/d),$$

where we have used the following estimate

$$\sum_{0\leqslant r\leqslant k} (-1)^{k-j} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d)(x/d)^{\ell/2} P_r\big(\log(x/d)\big) = x^{\ell/2} G_k^*(\log x) + O\big(x^{(\ell+2)/4+\varepsilon}\big).$$

Comparing with (2.3) of Lemma 2.2 yields

$$x^{\ell/2-1}\Delta_k(x) = \sum_{0\leqslant r\leqslant k} (-1)^{k-r} A_\ell^{-k} C_k^r \sum_{d\leqslant x} c_{k,r}(d) \Delta_r^*(Q, x/d) + O\big(x^{\ell/2-1+\delta_k}\big).$$

Under hypothesis (1.11), by (2.5), (2.3) of Lemma 2.2 and (2.7) of Lemma 2.3 we have

$$\begin{aligned}\Delta_r^*(Q, x) &\ll x^{\ell/2-1+r/\{2(r+1)\}-\delta+\varepsilon}\\ &\ll x^{\ell/2-1+(k-1)/(2k)-\delta/2}\end{aligned}$$

for $0\leqslant r\leqslant k-1$. Inserting into the preceding formula and using (2.7), we can deduce

$$x^{\ell/2-1}\Delta_k(x) = A_\ell^{-k} \sum_{d\leqslant x} c_{k,k}(d) \Delta_k^*(Q, x/d) + O\big(x^{\ell/2-1+(k-1)/(2k)-\delta/2}\big). \tag{4.1}$$

On the other hand, according to Soundararajan [10], we have, for any $k\geqslant 2$,

$$\Delta_k(x) = \Omega\big((x\log x)^{(k-1)/(2k)}(\log_2 x)^{\beta_k}(\log_3 x)^{-\gamma_k}\big). \tag{4.2}$$

Now on noting (2.7) of Lemma 2.3, the first assertion of Theorem 2 follows from (4.1) and (4.2).

Finally in view of (1.8), it is easy check that the hypothesis (1.11) is satisfied when $k = 2$ or 3. Therefore (1.12) holds unconditionally for these two values of k.

This completes the proof of Theorem 2. $\square$

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